Anisotropic Electron Distribution and the dc and Microwave Avalanche Breakdown in Hydrogen

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The Boltzmann transport equation is solved taking into account vibrational, dissociation, excitation, and ionization losses in hydrogen at E/p_0 between 40 and 450 V/cm-mm Hg. No approximations are made regarding the angular dependence of the electron distribution function. At the high E/p_0 the distribution function is sufficiently anisotropic that it cannot be represented by a two-term expansion in spherical harmonics. It is shown that the concept of "effective" field in microwave breakdown remains valid in the presence of anisotropies provided the circular frequency is much larger than the electron growth rate. The temporal growth rate of the electron density is calculated and compared with the experiments of Rose and of Cottingham and Buchsbaum. The effects of anisotropy on the diffusion coefficient and on the microwave breakdown equation are discussed.

I. INTRODUCTION

THE exponential growth of electron density in the microwave breakdown of hydrogen gas has been studied by Cottingham and Buchsbaum[1] under conditions which allow the gas to be regarded as being of infinite spatial extent. They obtained values of the growth rate β, $[N=N_0 \exp(\beta t)]$, for a range of E_e/p_0, the ratio of the effective electric field intensity to the pressure,[2] from 36 to 200 V/cm-mm Hg. The conditions of the experiment facilitate comparison of the results with the theory of Allis and Brown[3] and with the theory of Pearlstein and Stuart.[4] In both cases the agreement is poor. The microwave measurements do agree well with dc measurements by Rose[5] in which the Townsend-alpha coefficient (exponential spatial growth rather than temporal growth) was observed. The disagreement with the two theories is to be expected for reasons which will be apparent immediately; the agreement between the microwave measurements and the dc measurements has interesting consequences regarding the validity of the "effective-field" concept at high E_e/p_0.

The disagreement with the theory of Allis and Brown is to be expected because the theory, concerned primarily with low values of E_e/p_0, contains approximations which become less valid as E_e/p_0 is raised. One set of approximations is connected with the energy dependence of the various collision frequencies. Allis and Brown took these to be linearly increasing functions of energy. Also, Engelhardt and Phelps[6] have recently concluded that there is an appreciable probability that low-energy electrons will excite vibrational states of the hydrogen molecule. In order to obtain agreement with the microwave breakdown data, Allis and Brown had to disregard this process in their calculations.

Another approximation in the Allis and Brown theory is that the angular dependence of the electron distribution function is described by the first two terms in the spherical-harmonics expansion. On the basis of this approximation, they show that the microwave field may be replaced by an appropriately defined dc field, the so-called "effective" electric field. Our solution of the Boltzmann equation for the distribution function indicates that this angular approximation breaks down at high E/p_0, and that the distribution function becomes sufficiently anisotropic that it cannot be adequately described by only two spherical harmonics. One then wonders whether the effective electric field concept loses its validity at high E_e/p_0. The agreement between Cottingham's microwave measurements and Rose's dc measurements suggests strongly that in this case, at least, the effective electric field concept does hold. This serves to raise the question of whether the distribution function is really as anisotropic as our calculations suggest, or, whether the effective field concept is valid independently of the angular dependence of the distribution.

Finally, if we assume that the effective electric field concept is valid, then Cottingham's measurements can be compared with the theory of Pearlstein and Stuart,[4] which, without use of any angular approximations, predicts the growth rate for a dc discharge in an infinite medium. The lack of agreement here suggests that the Pearlstein-Stuart theory, essentially concerned with extremely high E/p_0 contains approximations which lose their validity at lower values of this parameter.[7]

The purpose of this paper is threefold. First, we show that at high E/p_0, the distribution function is sufficiently anisotropic that it cannot be represented by a two-term expansion in spherical harmonics. Second, we prove that the effective field concept is valid even at

[1] W. B. Cottingham and S. J. Buchsbaum, preceding paper [Phys. Rev. 130, 1002 (1963)].

[2] The effective electric field intensity E_e represents an equivalent dc field in the plasma. See H. Margenau, Phys. Rev. 69, 508 (1946). See also, reference 3.

[3] W. P. Allis and S. C. Brown, Phys. Rev. 87, 419 (1952).

[4] L. D. Pearlstein and G. W. Stuart, Phys. Fluids 4, 1293 (1961).

[5] D. J. Rose, Phys. Rev. 104, 273 (1956).

[6] A. G. Engelhardt and A. V. Phelps, Bull. Am. Phys. Soc. 7, 637 (1962).

[7] The main concern of Pearlstein and Stuart was to obtain upper bounds on the growth rate β as a function of E/p using simple models amenable to rapid calculations. Their treatment of the energy loss is compatible with these objectives, and their values of β certainly lie well above the measured values. In this sense the experimental result may be cited as support of the Pearlstein and Stuart theory.

high E/p_0, that is, in the presence of large anisotropies. Third, we calculate the growth rates as a function of E/p_0 and compare the calculated rates with experiment. We also point out the tensor nature of the diffusion coefficient at high E/p_0 and the need for modifying the diffusion-controlled breakdown equation when the distribution function is highly anisotropic.

The structure of the paper is as follows: In Sec. II we discuss the data on various collision frequencies which we use in the theory and the assumptions we make where data are lacking. Section III contains the derivation of the equations from which the distribution function and growth rates are calculated in the dc avalanche breakdown. Our method here is an extension of the approach initiated by Stuart and Gerjuoy.[8] We obtain a pair of equations whose simultaneous solution gives the spherically symmetric part of the distribution function and the exponential growth rate. From the spherically symmetric part of the distribution function it is possible to regain its full angular dependence. In this section we deviate from the procedure of Gerjuoy and Stuart[8] by introducing the inelastic collisions. We repeat many of the steps of reference 8 because in the derivation of the effective field in the microwave breakdown (Sec. IV) we need to establish contact with some of the intermediate equations of the dc case. Our final expression for the calculation of β differs somewhat from that used by Stuart and Gerjuoy. Finally, in Sec. V, we present and discuss results of the detailed calculations using the data of Sec. II.

II. BASIC CROSS SECTIONS AND ASSUMPTIONS

Let ν_e, ν_v, ν_x, ν_d, ν_i, and ν be the elastic, vibration, excitation, dissociation, ionization, and total collision
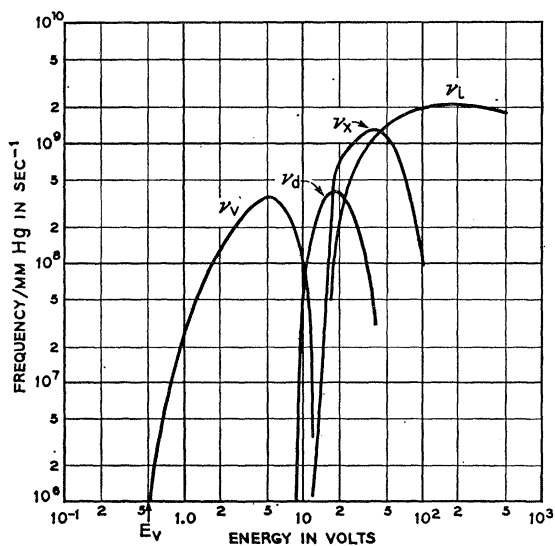

FIG. 1. Electron collision frequencies in H_2 vs energy for vibrational excitation, dissociation, electronic excitation, and ionization.

[8] G. W. Stuart and E. Gerjuoy, Phys. Rev. **119**, 892 (1960); E. Gerjuoy and G. W. Stuart, Phys. Fluids **3**, 1008 (1960).
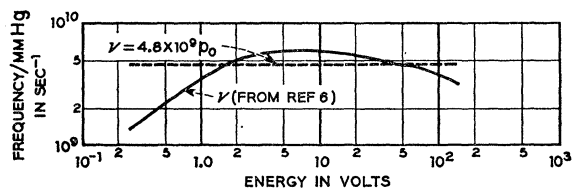

FIG. 2. Total collision frequency ν/p vs energy. The solid curve is from reference 6; the dashed line represents the value $4.8\times10^9 p_0$ as used in the present work.

frequencies, respectively. They are functions of the speed C of the electron; but for electron energies above about 4 eV, the total collision frequency ν depends only weakly on the speed. Hence, we shall take

$$\nu = \nu_e(C) + \nu_v(C) + \nu_x(C) + \nu_d(C) + \nu_i(C) \qquad (2.1)$$

to be a constant, independent of the electron speed and shall regard $\nu_e(C)$ as being calculated by means of (2.1). The quantities on the right-hand side of Eq. (2.1) have recently been determined by Engelhardt and Phelps[6] who give sources for their data and express their results as cross sections for collision. The corresponding collision frequencies for the last four processes on the right-hand side of Eq. (2.1) are shown plotted in Fig. 1. For the total collision frequency we use the value

$$\nu = 4.8 \times 10^9 p_0, \qquad (2.2)$$

where p_0 is the pressure in mm Hg reduced to 0°C of the neutral hydrogen gas. This value of ν results in best agreement between the microwave and dc measurements of the ionization frequency.[1] The total collision frequency ν as given by Eq. (2.2) is plotted in Fig. 2, where it may be compared with the corresponding frequency of reference 6.

We assume that the scattering is isotropic for each of the processes listed in Eq. (2.1), but that each process may have a different effect on the energy of the colliding electron. We list here our assumptions about the energy losses associated with each process and defer till later the justification of these assumptions.

(a) Elastic collisions leave the electron energy unchanged.

(b) Vibrational collisions decrease the electron energy by $E_v = 0.516$ eV, the threshold for vibrational energy transfer.

(c) Excitation, dissociation, and ionization result *either* in maximum energy loss, leaving the electron with zero energy, *or* in minimum energy loss, leaving the electron with its original energy reduced by the threshold energy for the process in question.

(d) Ionizing collisions release another zero-energy electron into the system.

When there is no need to distinguish between the various processes which leave an electron with most of its energy or between the various processes which return it to zero energy, we shall refer to the vibrational collisions as elastic, or near elastic, and shall refer to the remaining three processes as inelastic.

III. THE dc AVALANCHE BREAKDOWN

In this section, we consider the Boltzmann equation governing the electron distribution function for the case of the dc field. Initially, there is a single electron of zero energy in the system. It accelerates under the influence of the field and collides elastically with the gas molecules until it achieves sufficient energy to suffer an inelastic collision which returns it to nearly zero energy. If the inelastic collision is an ionizing one, a second electron of zero energy is released. After the collision, the newly released electrons continue the cycle of acceleration and collision. The problem is to calculate the rate of growth of the total number of electrons in the system and their distribution in energy and direction.

The history of the electrons in the distribution may be divided naturally into intervals between successive inelastic collisions, because in each such period, the time development of the distribution function of the electrons released at, or returned to, zero energy is the same as that of the original electron during the first acceleration period. This point of view will appear early in the mathematical description. It is an extension of the idea contributed by Gerjuoy and Stuart[8] in their theory of dc breakdown of a gas in which all collisions were assumed to be elastic. We shall also follow Gerjuoy and Stuart in performing the angular integration of the Boltzmann equation so as to obtain an integral equation for the angular average of the distribution function. This frees us from the necessity of making approximations about the angular dependence. Having found the angular average, we shall be able to compute from it the angular dependence of the distribution function.

The distribution function $f(\mathbf{v},t)$, which gives the number of electrons in the velocity element around $\mathbf{v}$ at time t, is governed by the following Boltzmann equation and initial conditions:

$$\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial v_z}+\nu\right)f(\mathbf{v},t)$$

$$=\frac{1}{4\pi C^2}\int d\mathbf{v}'\,f(\mathbf{v}',t)[\nu_e(C')\delta(C-C')$$

$$+\nu_v(C')\delta(C-C'')]+S(t)\delta(C)/(4\pi C^2);\quad (3.1\text{a})$$

$$S(t)=\int d\mathbf{v}\,[\nu_x(C)+\nu_d(C)+2\nu_i(C)]f(\mathbf{v},t);\quad (3.1\text{b})$$

$$f(\mathbf{v},0)=\delta(C)/(4\pi C^2);\quad (3.1\text{c})$$

$$C=|\mathbf{v}|,\quad C'=|\mathbf{v}'|,\quad C''=[(C')^2-2E_v/m]^{1/2}.\quad (3.1\text{d})$$

Here, $\mathbf{a}$ is the acceleration caused by an electric field directed towards negative z, and E_v is the fixed vibrational energy loss. The term $\nu f(\mathbf{v},t)$ on the left-hand side of Eq. (3.1a) represents the scattering of electrons out of the velocity interval $d\mathbf{v}$ at $\mathbf{v}$. The first term on the right of Eq. (3.1a) represents elastic scattering into this velocity interval from all other velocities with the same energy. The second term on the right of Eq. (3.1a) represents scattering into this velocity range by vibrational processes, from all velocities for which the energy is greater than $mv^2/2$ by an amount E_v, the threshold for vibrational excitation. The third term on the right of Eq. (3.1a) represents the inelastic collisions which supply electrons at a rate $S(t)$ to the velocity element at *zero energy*. This rate of supply is given in (3.1b)—one for each excitation and dissociation, two for each ionization. Thus, Eq. (3.1) corresponds to the assumption of "maximum energy loss" on inelastic collision as discussed in Sec. II. A similar equation can be written to correspond to the "minimum energy loss" assumption. It can be solved by the same method as that used for Eq. (3.1). We shall only quote the results of such a parallel calculation, and will label them "minimum energy loss."

Neglected in Eq. (3.1a) is the $(2m/M)$ energy loss on elastic collision, where m and M are the masses of the electrons and hydrogen molecules, and the energy loss due to excitation of rotational degrees of freedom. These energy losses are indeed negligible over the whole range of E/p_0 that we shall consider.

The study of Eq. (3.1) is facilitated by taking its Laplace transform with respect to time. On defining

$$\tilde{f}(\mathbf{v},p)=\int_0^\infty dt\,e^{-pt}f(\mathbf{v},t),\quad (3.2)$$

$$\tilde{S}(p)=\int_0^\infty dt\,e^{-pt}S(t),\quad (3.3)$$

and making use of the delta function in the elastic collision terms, the Laplace transform of (3.1) becomes

$$\left(p+a\frac{\partial}{\partial v_z}+\nu\right)\tilde{f}(\mathbf{v},p)$$

$$=\frac{1}{4\pi}\int d\Omega\,[\nu_e(C)\tilde{f}(C\Omega,p)+(C'/C)\nu_v(C')\tilde{f}(C'\Omega,p)]$$

$$+[1+\tilde{S}(p)]\delta(C)/4\pi C^2;\quad (3.4\text{a})$$

$$C'=(C^2+2E_v/m)^{1/2};\quad \mathbf{v}'=C'\Omega;\quad (3.4\text{b})$$

$$\tilde{S}(p)=\int d\mathbf{v}\,[\nu_x(C)+\nu_d(C)+2\nu_i(C)]\tilde{f}(\mathbf{v},p).\quad (3.5)$$

We divide Eq. (3.4a) by $(1+\tilde{S})$ to show that there is a function $\tilde{g}(\mathbf{v},p)$ satisfying

$$\left(p+a\frac{\partial}{\partial v_z}+\nu\right)\tilde{g}(\mathbf{v},p)=\frac{1}{4\pi}\int d\Omega\,[\nu_e(C)\tilde{g}(C\Omega,p)$$

$$+(C'/C)\nu_v(C')\tilde{g}(C'\Omega,p)]+\delta(C)/4\pi C^2,\quad (3.6)$$

which is related to $\tilde{f}$ by means of

$$\tilde{f}(\mathbf{v},p) = [1+\tilde{S}(p)]\tilde{g}(\mathbf{v},p). \qquad (3.7)$$

Multiplying (3.7) by $\nu_x+\nu_d+2\nu_i$ and integrating yields

$$\tilde{S}(p) = [1+\tilde{S}(p)]\tilde{N}(p),$$

or

$$\tilde{S}(p) = \tilde{N}(p)[1-\tilde{N}(p)]^{-1}, \qquad (3.8)$$

where

$$\tilde{N}(p) = \int d\mathbf{v} \,[\nu_x(C)+\nu_d(C)+2\nu_i(C)]\tilde{g}(\mathbf{v},p). \quad (3.9)$$

Combining (3.8) and (3.7) gives

$$\tilde{f}(\mathbf{v},p) = [1-\tilde{N}(p)]^{-1}\tilde{g}(\mathbf{v},p), \qquad (3.10)$$

and the Laplace inversion of (3.10) yields

$$f(\mathbf{v},t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dp\, e^{pt}\frac{\tilde{g}(\mathbf{v},p)}{1-\tilde{N}(p)}. \qquad (3.10a)$$

The long time behavior of $f(\mathbf{v},t)$ is controlled by the singularity of the integrand furthest to the right in the complex p plane. In Appendix A, we show that this singularity is a simple pole, located at $p=\beta$, where

$$\tilde{N}(\beta) = 1. \qquad (3.11)$$

Further, we show that Eq. (3.11) always has a unique positive real solution for β and that any solution of (3.11) for β off the real axis necessarily lies to the left of the real solution. This means that the long time behavior of $f(\mathbf{v},t)$ is

$$f(\mathbf{v},t) \propto \tilde{g}(\mathbf{v},\beta)e^{\beta t}, \qquad (3.12)$$

where β is the real positive solution of (3.11). Thus the experimentally observable exponential growth rate may be calculated from Eq. (3.11). This is the procedure followed in reference 8. Actually, Eq. (3.11) is not convenient for numerical calculations, partly because it is implicit in β, and partly because its solution is very sensitive to the normalization of $\tilde{g}$. Both drawbacks may be overcome as follows: Consider the integral of (3.6) over all velocities. Setting $p=\beta$ and making use of (2.1) gives

$$\int d\mathbf{v} \,[\beta+\nu_x(C)+\nu_d(C)+\nu_i(C)]\tilde{g}(\mathbf{v},\beta) = 1. \quad (3.13)$$

Now consider Eq. (3.11), using the definition (3.9)

$$\int d\mathbf{v} \,[\nu_x(C)+\nu_d(C)+2\nu_i(C)]\tilde{g}(\mathbf{v},\beta) = 1. \quad (3.14)$$

Subtracting these two equations and solving for the explicit β gives

$$\beta = \int d\mathbf{v}\, \tilde{g}(\mathbf{v},\beta)\nu_i(C) \Big/ \int d\mathbf{v}\, \tilde{g}(\mathbf{v},\beta), \qquad (3.15)$$

which is independent of the normalization of $\tilde{g}$. Equation (3.15) is analogous to that used in conventional calculations of β,[3] that is, in calculations which start from the steady-state Boltzmann equation in which the gain in particle density through ionization is exactly balanced by loss of particles through diffusion or attachment (the breakdown equation). If $G(\mathbf{v})$ is the solution of such a steady-state Boltzmann equation, it takes place of $\tilde{g}(\mathbf{v},\beta)$ in (3.15) in a conventional calculation. Note that $G(\mathbf{v}) \neq \tilde{g}(\mathbf{v},0)$ but is more closely related to $\tilde{g}(\mathbf{v},\beta)$. This is because Eq. (3.6) can be interpreted as a steady-state equation in which there is a constant collision frequency for particle loss. This frequency p must be set equal to β to yield $\tilde{g}(\mathbf{v},\beta)$. Equating p, the frequency for particle loss, to β, the exponential growth rate in our theory is precisely analogous to the assumptions of a steady-state breakdown equation; growth of particle density by ionization is equal to loss due to diffusion and attachment.

Having found a convenient relation between β and $\tilde{g}$, we turn our attention now to the equation determining $\tilde{g}$. We rewrite Eq. (3.6) with p set equal to β, as

$$(\beta+a\partial/\partial v_z+\nu)\tilde{g}(\mathbf{v},\beta) = \nu Q(C)/4\pi; \qquad (3.16a)$$

$$Q(C) = \frac{\delta(C)}{(\nu C^2)} + \int d\Omega \Big[r_e(C)\tilde{g}(C\mathbf{\Omega},\beta)$$

$$+ \Big(\frac{C'}{C}\Big)r_v(C')g(C'\mathbf{\Omega},\beta) \Big]; \qquad (3.16b)$$

$$r_e(C) = \nu_e(C)/\nu; \quad r_v(C) = \nu_v(C)/\nu. \qquad (3.16c)$$

Equation (3.16) may be integrated readily to give

$$\tilde{g}(\mathbf{v},\beta) = \frac{\nu}{4\pi a} \int_{-\infty}^{v_z} dw\, e^{-(v_z-w)(\beta+\nu)/a}Q[(v_x^2+v_y^2+w^2)^{1/2}].$$

A change of variables to spherical coordinates

$$v_z-w = R,$$
$$v_x = C \sin\theta \sin\varphi,$$
$$v_y = C \sin\theta \cos\varphi,$$
$$v_z = C \cos\theta = C\mu,$$

yields

$$\tilde{g}(\mathbf{v},\beta) = \frac{\nu}{4\pi a} \int_0^\infty dR\, e^{-R(\beta+\nu)/a}Q(C''), \qquad (3.17a)$$

$$C'' = (C^2-2\mu RC+R^2)^{1/2}. \qquad (3.17b)$$

Finally, the angular integration may be performed,

$$\int \tilde{g}d\Omega = \frac{\nu}{2a} \int_{-1}^{1} d\mu \int_0^\infty dR\, e^{-R(\beta+\nu)/a}Q(C''). \quad (3.18)$$

Using C'' rather than μ as a variable of integration in

Eq. (3.17), yields[8]

$$\int \tilde{g}d\Omega = \int_0^\infty \left(\frac{C''}{C}\right)^2 K(C,C'')Q(C'')dC'', \quad (3.19a)$$

where

$$K(C,C'') = \frac{\nu}{2a}\left(\frac{C}{C''}\right)\left[E_1\left(|C-C''|\frac{\beta+\nu}{a}\right)\right.$$

$$\left. -E_1\left(|C+C''|\frac{\beta+\nu}{a}\right)\right], \quad (3.19b)$$

and

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt, \quad (3.19c)$$

is the tabulated exponential integral function. It is convenient to define

$$\rho(\beta,C) \equiv C^2 \int d\Omega\, \tilde{g}(\mathbf{v},\beta), \quad (3.20a)$$

so that, by use of (3.16b) and (3.20a) the integral equation (3.19) can be rewritten as

$$\rho(\beta,C) = a^{-1}e^{-(\beta+\nu)C/a}$$

$$+\int_0^\infty dC'' K(C,C'')[r_e(C'')\rho(\beta,C'')$$

$$+(C/C^*)r_v(C^*)\rho(\beta,C^*)], \quad (3.20b)$$

$$C^* = [(C'')^2 + 2E_v/m]^{1/2}. \quad (3.20c)$$

The inhomogeneous term in (3.20b) arises in taking the limit $K(C, C'' \to 0)$ which is implied by the $\delta(C'')$ appearing in (3.16b). In terms of ρ, Eq. (3.15) for β becomes

$$\beta/\nu = \int_0^\infty dC\, \rho(\beta,C)r_i(C) \left/ \int_0^\infty dC\, \rho(\beta,C),\right. \quad (3.21a)$$

$$r_i(C) = \nu_i(C)/\nu. \quad (3.21b)$$

The difference between Eq. (3.20b) and the analogous equation of reference 8 is the presence of r_v and r_e in the integrand. Without these factors, or, in the special case in which r_e were independent of C' and r_v were zero, the solution of Eq. (3.20b) could be obtained using the methods of reference 8. The factors arise, of course, because we wish to allow inelastic collisions to remove electrons from the speed range at C'.

Equations (3.20b) and (3.21) must be solved simultaneously. For this purpose, a new set of variables is useful:

$$x = C/\gamma, \quad (3.22a)$$

$$y = C''/\gamma, \quad (3.22b)$$

$$\gamma = a/(\beta+\nu), \quad (3.22c)$$

$$\eta = (1+\beta/\nu)^{-1}, \quad (3.22d)$$

$$N(x,\eta) \equiv a\rho(\beta,C). \quad (3.22e)$$

The equations to be solved become

$$N(x,\eta) = e^{-x} + \frac{\eta}{2}\int_0^\infty dy\left(\frac{x}{y}\right)[E_1(|x-y|)-E_1(|x+y|)]$$

$$\times\{r_e(\gamma y)N(y,\eta)+(y/y^*)r_v(\gamma y^*)N(y^*,\eta)\},$$

$$(3.23a)$$

$$y^* = [y^2+2E_v/m\gamma^2]^{1/2}, \quad (3.23b)$$

$$1/\eta = 1 + \int_0^\infty dx\, N(x,\eta)r_i(\gamma x) \left/ \int_0^\infty dx\, N(x,\eta)\right. \quad (3.23c)$$

Equations (3.23) must be handled numerically. The solution may be obtained by iteration. Having chosen a value for γ, one chooses a trial form for $N(x,\eta)$ in (3.23c) and computes η. The computed η and trial form of N are used in (3.23a) to obtain an improved form for N. The improved form is then used in (3.23c) again and the cycle repeated until η converges.[9] Having obtained η for the chosen value of γ, one obtains via (3.22d) and (3.22c)

$$\beta/\nu = 1/\eta - 1, \quad (3.24a)$$

$$a/\nu = \gamma/\eta. \quad (3.24b)$$

By choosing other values for γ and repeating the process, a plot of β/ν vs a/ν may be constructed. Since ν is directly proportional to the pressure, the plot can be expressed in the conventional units of β/p_0 vs E/p_0.

Finally, having obtained $N(x,\eta)$, one evaluates $Q(C)$, Eq. (3.16b), by means of (3.20a) and (3.22e). From $Q(C)$, the angular-dependent distribution function may be obtained by performing the integration indicated in Eq. (3.17a).

IV. MICROWAVE AVALANCHE BREAKDOWN

We shall now consider the time-dependent solution of the Boltzmann equation in the presence of a high-frequency electric field. Our aim is to establish the validity of the "effective" field concept without recourse to an expansion of the distribution function in spherical harmonics. Our technique here will be to separate the time dependence arising from the harmonic time dependence of the field direction from the slower time dependence arising from the transfer of energy from the field to the electrons (and to the gas) and the resulting growth of the electron density. The functions describing the slower time dependence will prove to be the

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[9] It can be shown that such iterative procedure converges provided $\eta r_e(C) < 1$; this is obviously so in this problem.

analogues of the function which appeared in the dc treatment. We carry the study of the high-frequency case only so far as to establish contact with such intermediate equations of the previous section which will allow a proper definition of the "effective" field to be made. Our steps and notations duplicate most of the steps and notations of the previous section although many operations which previously were performed on Laplace transforms will have to be performed on the time-dependent functions. We shall set ν_v and ν_d equal to zero in order to simplify the notation; it will be quite evident, from the close parallel with Sec. III, just how the equations can be modified to include these processes. Such modifications do not affect the validity of any step in this section.

The time-dependent Boltzmann equation and initial conditions that describe the electron distribution resulting from the release of a single zero-energy electron into an infinite system in the presence of a high-frequency electric field are

$$\left(\frac{\partial}{\partial t}+a\,\cos\omega t\frac{\partial}{\partial v_z}+\nu\right)f(\mathbf{v},t)$$

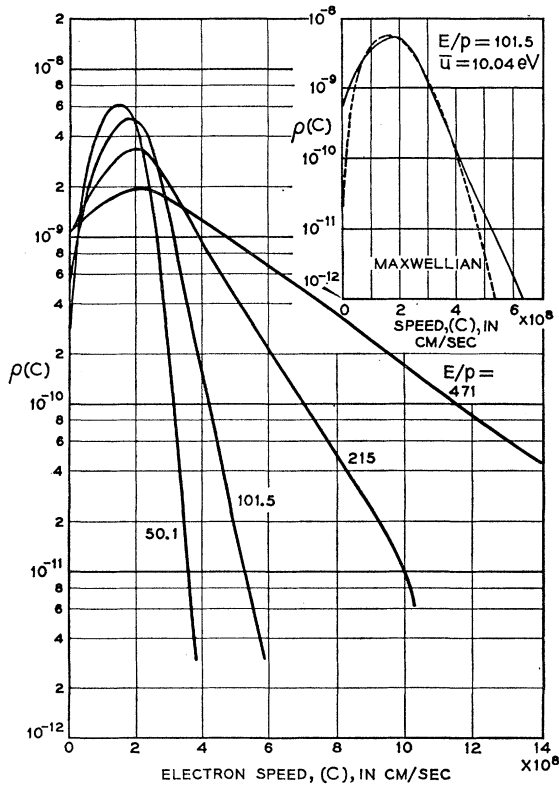$$=\frac{1}{4\pi}\int d\Omega\,\nu_e(C')f(\mathbf{v}',t)+S(t)\delta(C)/(4\pi C^2),\quad (4.1a)$$

FIG. 3. Electron distribution function $\rho(C)$ vs electron speed calculated assuming "minimum energy loss."

$$S(t)=\int d\mathbf{v}\,[\nu_x(C)+2\nu_i(C)]f(\mathbf{v},t),\quad (4.1b)$$

$$f(\mathbf{v},t=0)=\delta(C)/(4\pi C^2).\quad (4.1c)$$

We introduce the Green's function for this equation, $g(\mathbf{v},t,t')$, which satisfies

$$\left(\frac{\partial}{\partial t}+a\,\cos\omega t\frac{\partial}{\partial v_z}+\nu\right)g(\mathbf{v},t,t')$$

$$=\frac{1}{4\pi}\int d\Omega\,\nu_e(C)g(\mathbf{v}',t,t'),$$

$$g(\mathbf{v},t',t')=\delta(C)/(4\pi C^2).\quad (4.2)$$

Then direct substitution shows that

$$f(\mathbf{v},t)=g(\mathbf{v},t,0)+\int_0^t dt'\,g(\mathbf{v},t,t')S(t').\quad (4.3)$$

Equation (4.3) is, of course, the analog of Eq. (3.7). We proceed in much the same way as in Sec. III, multiplying (4.3) by $\nu_x+2\nu_i$, integrating over $\mathbf{v}$ and using (4.1b) to obtain

$$S(t)=N(t,0)+\int_0^t dt'\,N(t,t')S(t'),\quad (4.4)$$

$$N(t,t')=\int d\mathbf{v}\,[\nu_x(C)+2\nu_i(C)]g(\mathbf{v},t,t').\quad (4.5)$$

Now *suppose* that the times t and t' were to enter Eqs. (4.4) and (4.5) only through their difference, $(t-t')$. Then these equations would be exactly analogous to Eqs. (3.8) and (3.9) of Sec. III because on taking Laplace transforms, Eqs. (4.4) and (4.5) would become

$$\tilde{S}(p)=[1+\tilde{S}(p)]\tilde{N}(p),\quad (4.6)$$

$$\tilde{N}(p)=\int d\mathbf{v}\,[\nu_x(C)+2\nu_i(C)]\tilde{g}(\mathbf{v},p).\quad (4.7)$$

The same line of reasoning as was used in Sec. III would again lead to the conclusion that the density grows exponentially with a growth rate β determined by

$$N(\beta)=1.$$

There would then remain the problem of calculating $\tilde{N}(p)$. If we could show that $\tilde{N}(p)$ corresponds to the analogous function in the preceding section, the equivalence between the microwave case and the dc case would be established.

We show in Appendix B that under certain conditions an $\tilde{N}(p)$ can be defined which is analogous to the corresponding function in dc-field case. These conditions require that the circular frequency ω of the microwave

field be larger than the growth rate β and that the amplitude of the microwave field be not so large that the first collision is likely to be inelastic. When these conditions are satisfied and the correspondence is possible, there emerges an effective electric field given by

$$E_e = \frac{1}{\sqrt{2}} \frac{E}{[1+\omega^2/(\nu+\beta)^2]^{1/2}},\qquad (4.8)$$

which plays the same role in the microwave growth rate equation as does the dc electric field in Sec. III.

The equivalence between the microwave and dc cases goes further than the growth rates, of course. We also show in Appendix B, that a function $\bar{g}^0(v,\beta)$, which gives the time dependence of the slowly varying part of distribution, also exists, and is obtained by averaging the dc distribution function over the directions in which the microwave field points [Eq. (B17)].

V. RESULTS AND CONCLUSIONS

We have evaluated the distribution function $g(v,\beta)$ and the coefficient β for a range of E/p_0 from 40 to 450 V/cm-mm Hg using the two assumptions of "minimum" and "maximum energy loss" of Sec. II. We also evaluated the average energy $\bar{u}$ and the diffusion coefficient D.

The Distribution Function

In Figs. 3 and 4 is plotted the function $\rho(\beta,C)$ [Eq. (3.20a)] as function of the speed C for various values of E/p_0 and for the two assumptions of "minimum" and "maximum energy loss." The function is so normalized that $\int_0^\infty \rho(C)dC = 1$. The presence of electrons at zero speed in $\rho(C)$ is, of course, a consequence of our assumption of the appearance of such electrons as a result of inelastic collisions. It is interesting to compare $\rho(C)$ with a suitably normalized Maxwellian distribution of the same average energy; such a representative comparison is shown in the inserts in Figs. 3 and 4 for an E/p_0 of 100 V/cm-mm Hg. It can be seen that at high speeds $\rho(C)$ falls off less steeply than the Maxwellian, the effect being more pronounced for the "maximum-energy-loss" assumption. Thus, an experimental measurement of the dependence of the tail of the distribution on energy, generally will lead to an overestimation of the average energy of the distribution. Such effect was observed by Whitehouse.[10]

The plots of Fig. 5 depict the anisotropy of distribution function for various values of E/p and for various speeds at a given E/p. These are polar plots, the radius vector from the origin to any point on the curve being proportional to the number of electrons traveling at the given speed in the direction that the radius vector makes with the direction of acceleration. The scale on

[10] D. R. Whitehouse, Ph.D. dissertation, Department of Electrical Engineering, Massachusetts Institute of Technology, 1957 (unpublished).
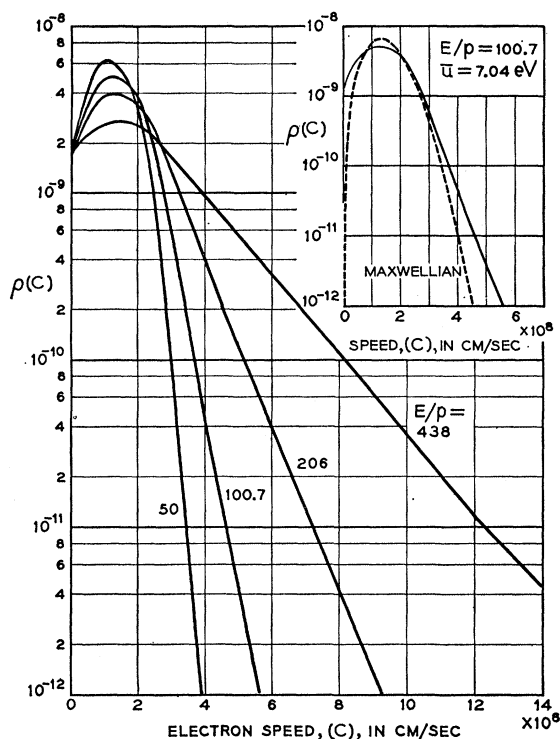

FIG. 4. Electron distribution function $\rho(C)$ vs electron speed calculated assuming "maximum energy loss."

these plots is relative; for a given E/p the scale is the same for all speeds but changes for various E/p's in order that all plots be of approximately the same physical size.

The anisotropy has two forms: a "spike" in the direction of the acceleration and a "flattening" of the over-all distribution into an elliptical shape. The "spike" is made up of those "fortunate" electrons which have eluded collisions and which continue to travel along the direction of the acceleration. The number of electrons within the "spike" is very small and decreases with electron speed. The flattening of the distribution is very pronounced at high E/p and survives even at electron energies of the order of the excitation and ionization energies. As expected, it is larger for the "maximum" than for the "minimum energy loss" assumption. This pronounced flattening is the reason for failure of the spherical harmonics expansion to properly represent the distribution function with only two terms in the expansion. Since low-speed electrons are more anisotropic than high-speed electrons, the spherical harmonic expansion will fail at even lower E/p for calculation of the transport coefficients (such as diffusion) which depend more strongly on low-speed electrons.

The Average Energy and Growth Rate

In Fig. 6 is plotted the growth-rate β as a function of p_0/E_e. The experimental results of Rose and of Cotting-

ham are shown as a dashed curve. It is gratifying that the measured values fall between the two calculated curves.

Not much weight is attached to the fact that the experimental curve is closer to the "maximum energy loss" curve than to the "minimum energy loss" curve. It must be remembered that the cross sections as quoted in Sec. II are (with the exception of the ionization cross section) not measured cross sections, but are deduced by Engelhardt and Phelps by fitting theoretical calculations to transport coefficients measured at low

E/p_0. The resulting set of cross sections, although it is realistic, is admittedly not unique. We also remind the reader that we assumed that in an ionizing collision an electron is born with zero energy regardless of whether the initial electron loses the "minimum" or "maximum" energy. Actually, the energy excess over the ionization threshold is shared by the two emergent electrons. This will tend to raise the "maximum energy loss" curve in Fig. 6 and lower the "minimum energy loss" curve, and thus reduce the spread between the two.

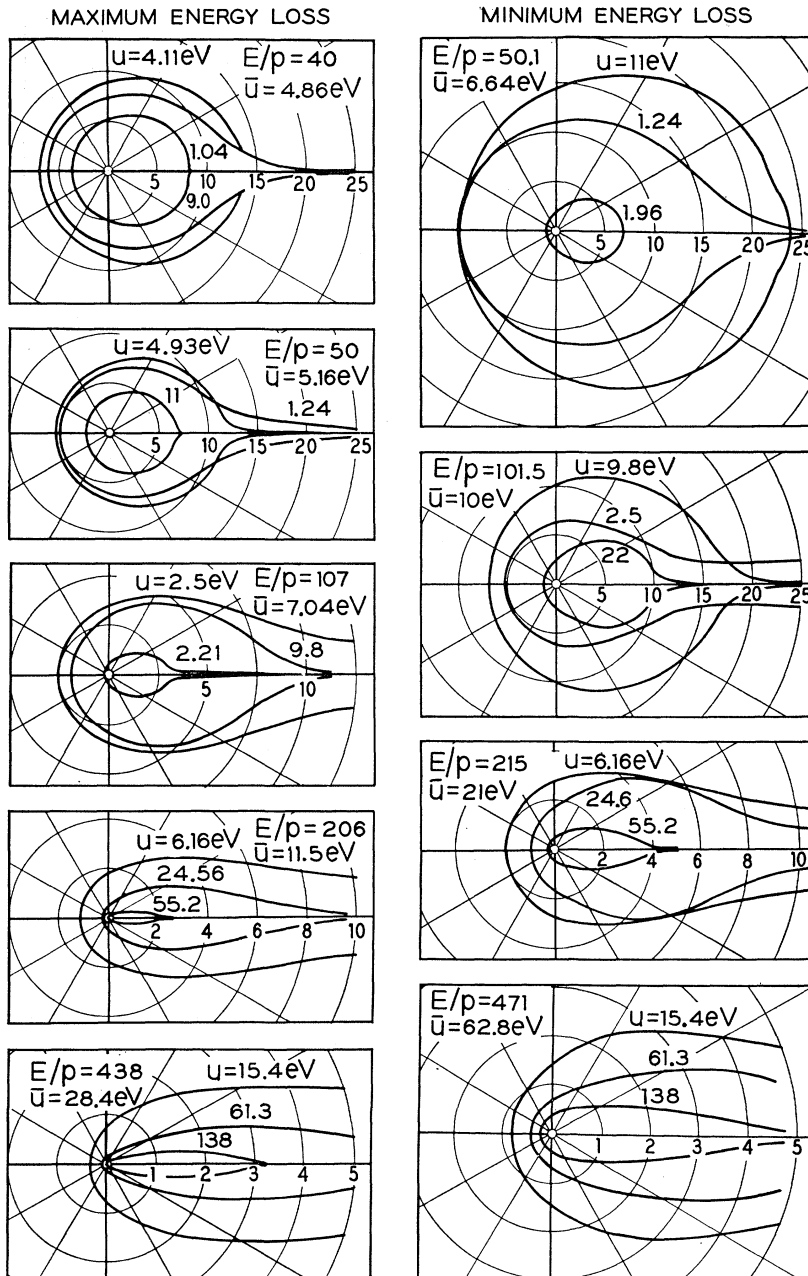In Fig. 7 is plotted the average energy $\bar{u}$ as a function

FIG. 5. Polar plots of the electron velocity distribution for various values of E/p and the coresponding average energy $\bar{u}$. The energy u for which the angular distributions are plotted are indicated.

of p_0/E_e. The average energy increases rapidly with E_e/p_0 for E_e/p_0 greater than about 200 V/cm-mm Hg, more rapidly in fact than the growth rate β. The rapid increase of $\bar{u}$ results from the fact that at very high E_e/p_0 the electrons are able to "punch" their way in energy past the inelastic sinks of energy.

The Diffusion Coefficient

Because the distribution function is highly aniso-tropic at large E/p_0, the diffusion coefficient

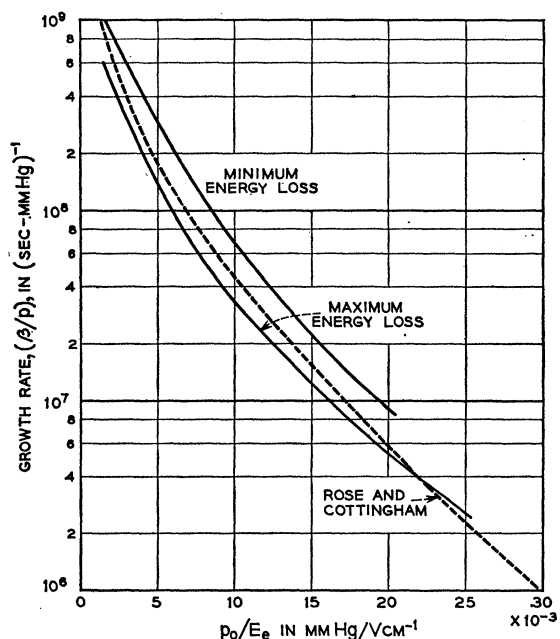$$\mathbf{D} = (1/\nu)\int \mathbf{vv} f(C,\mu) d^3 v \qquad (5.1)$$

FIG. 6. Plot of the growth rate β vs p_0/E_e; the dashed curve represents the measured values of β obtained by Rose and by Cottingham and Buchsbaum.

is no longer a scalar, but is a diagonal tensor with components D_{11}, $D_\perp$, $D_\perp$, where the subscripts "$\parallel$" and "$\perp$" denote the components along and perpendicular to the direction of the applied electric field. By expanding f in Legendre polynomials in μ,

$$f(C,\mu) = \sum_l f_l(C) P_l(\mu), \qquad (5.2)$$

it can be easily shown that D_{11} and $D_\perp$ are rigorously given by

$$D_{11} = \tfrac{2}{3}\bar{u}\mu_e[1+\tfrac{2}{5}\delta], \qquad (5.3a)$$

$$D_\perp = \tfrac{2}{3}\bar{u}\mu_e[1-\tfrac{1}{5}\delta], \qquad (5.3b)$$

$$\delta = \left[\int_0^\infty f_2(C)C^4 dC\right]\Big/\left[\int_0^\infty f_0(C)C^4 dC\right], \qquad (5.3c)$$
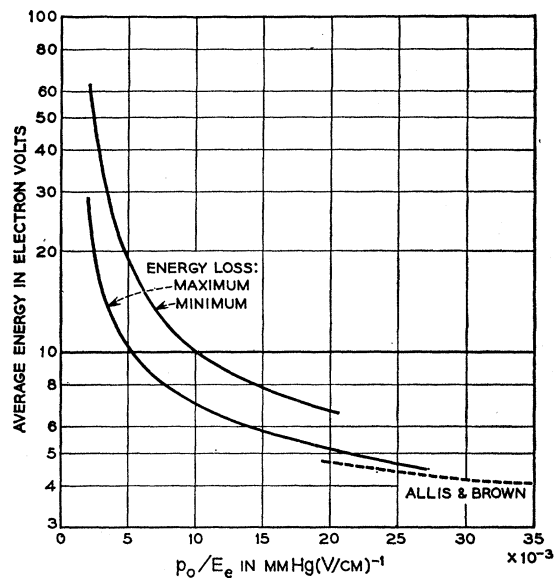
FIG. 7. Plot of the average energy vs p_0/E_e; the dashed curve is from the theory of Allis and Brown.

where $\bar{u}$ is the average energy in eV and $\mu_e = e/m\nu$ is the electron mobility. Note that D_{11} and $D_\perp$ separately no longer obey the Einstein relation.[11] The correction factor δ is shown in Fig. 8 as a function of p_0/E_e. At low E_e/p_0, δ tends to zero, while at high E_e/p_0, δ tends to its limiting value 5 which obtains for a needle-like distribution. Since δ is a measure of the importance of the second spherical harmonic in the expansion of f, an expansion in which only the terms f_0 and f_1 are retained must fail when δ becomes appreciable.
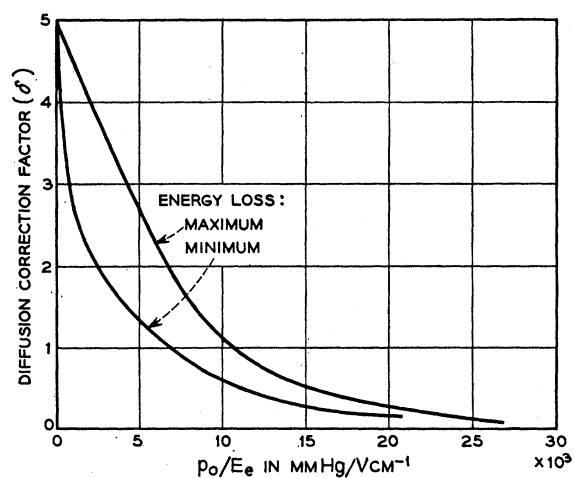
FIG. 8. Plot of the diffusion-coefficient correction factor vs p_0/E_e.

[11] W. P. Allis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, p. 414.

Breakdown

When the distribution function is nearly isotropic the diffusion-controlled microwave-breakdown condition is defined by the equation[12]

$$\beta = D/\Lambda^2, \tag{5.4}$$

where D is free-diffusion coefficient and Λ is diffusion length for the fundamental or lowest diffusion mode. From the Einstein relation $D = \frac{2}{3}\bar{u}\mu_e$, Eq. (5.4) can be written as

$$(p\Lambda)^2 = \frac{2}{3}(\mu_e p)\bar{u}/(\beta/p), \tag{5.5}$$

which exhibits the fact that the proper variable[12] $(p\Lambda)$ depends on (E_e/p) only.

As we have seen, in general D is a tensor, so that the more general breakdown equation must be written as

$$\beta = D_{11}/\Lambda_{11}{}^2 + D_{\perp}/\Lambda_{\perp}{}^2, \tag{5.6}$$

where Λ_{11} and $\Lambda_{\perp}$ are the lowest-mode diffusion lengths along and perpendicular to the direction of the breakdown field. Using Eq. (5.3) the breakdown equation can be written in the form

$$(p\Lambda)^2 = \frac{2}{3}\frac{\bar{u}(\mu_e p)}{(\beta/p)}\left[1 + \frac{1}{5}\delta\frac{2 - (\Lambda_{11}{}^2/\Lambda_{\perp}{}^2)}{1 + (\Lambda_{11}{}^2/\Lambda_{\perp}{}^2)}\right], \tag{5.7}$$

where

$$1/\Lambda^2 = 1/\Lambda_{11}{}^2 + 1/\Lambda_{\perp}{}^2. \tag{5.8}$$

Equation (5.7) indicates that at high E_e/p where δ is appreciable, the variable $(p\Lambda)$ depends not only on E_e/p but on the ratio $\Lambda_{11}/\Lambda_{\perp}$ as well, and thus on the shape of the container.

ACKNOWLEDGMENTS

APPENDIX A

Analytic Properties of the Transform

Consider the equation of which (3.6) is the Laplace transform.

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial v_z} + \nu\right)g(\mathbf{v},t) = \frac{1}{4\pi}\int d\Omega\left[g(C\mathbf{\Omega},t)\nu_e(C)\right.$$
$$\left. + \left(\frac{C'}{C}\right)\nu_v(C')g(C'\mathbf{\Omega},t)\right], \tag{A1}$$

$$g(v,t \to 0) = \delta(C)/(4\pi C^2), \tag{A2}$$

$$\tilde{g}(\mathbf{v},p) = \int_0^\infty dt\, e^{-pt}g(\mathbf{v},t), \tag{A3}$$

and the equation of which (3.9) is the Laplace transform.

$$N(t) = \int d\mathbf{v}\,[\nu_x(C) + \nu_d(C) + 2\nu_i(C)]g(\mathbf{v},t), \tag{A4}$$

$$\tilde{N}(p) = \int_0^\infty dt\, e^{-pt}N(t). \tag{A5}$$

Equations (A1) and (A2) indicate that $g(\mathbf{v},t)$ is the electron distribution in a gas which is similar to hydrogen, except that the excitation and ionization collisions in hydrogen become absorption collisions in this gas. This means that $g(\mathbf{v},t)$ is real and positive, and, because electrons can be absorbed, g will eventually decrease exponentially with time. From these properties, it follows solely because of (A3), that

(1) $\tilde{g}$ is analytic for all p with $\mathrm{Re}\,p \geq 0$, (A7)

(2) $\tilde{g}$ is real for p real and positive, (A8)

(3) $\tilde{g}$ is a monotonically decreasing function of p, for p real and positive. (A9)

Equations (A4) and (A5) indicate that these same properties apply to $\tilde{N}$ as well. Thus $\tilde{f}(\mathbf{v},p)$ in Eq. (3.10) can have a singularity at $\mathrm{Re}\,p \geq 0$ only where $\tilde{N}(p) = 1$. Properties (A7) and (A9), applied to N, limit the singularity of $\tilde{f}$ to be one or more simple poles. Property (A9) guarantees that there will be one and only one pole on the real axis if $\tilde{N}(0) \geq 1$. That this is the case may be verified by comparing (3.13), with β set equal to zero, with (3.9) with p set equal to zero. This comparison shows that

$$\tilde{N}(0) = 1 + \int d\mathbf{v}\,\nu_i(C)\tilde{g}(v,0) \geq 1.$$

The last point which must be established is that any solution of $N(p) = 1$ lies to the left of β. This will guarantee that $\exp\beta t$ is the dominant term at large t. To show that this is indeed the case, let $p = p_r + ip_i$ in (A5) and take the absolute value of this equation

$$|\tilde{N}(p)| = \left|\int_0^\infty e^{-p_r t}e^{-ip_i t}N(t)dt\right|$$

$$\leq \int_0^\infty |e^{-p_r t}|\,|e^{-ip_i t}|\,|N(t)|dt$$

$$= \tilde{N}(p_r). \tag{A10}$$

The equality holds only if the oscillating exponential is constant, i.e., if $p_i = 0$. Now if $\tilde{N}(p_r + ip_i) = 1$, then, by (A10), $\tilde{N}(p_r) > 1$ which, by (A9), means $p_r < \beta$.

[12] S. C. Brown, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 22, p. 531. See especially Sec. 4.

APPENDIX B

The Effective Field E_e

We wish to show that provided $\omega > \beta$, an effective field E_e can be defined which reduces the microwave-breakdown growth rate equations to those of the dc field case.

Define

$$\tau = t - t', \tag{B1a}$$

$$\varphi = \omega t', \tag{B1b}$$

and

$$g_\varphi(\mathbf{v}, \tau) = g(\mathbf{v}, t, t'). \tag{B1c}$$

Then Eq. (4.2) becomes

$$\left[\frac{\partial}{\partial \tau} + a \cos(\omega\tau + \varphi)\frac{\partial}{\partial v_z} + \nu\right] g_\varphi(\mathbf{v}, \tau)$$

$$= \frac{1}{4\pi} \int d\Omega \, \nu_e(C) g_\varphi(\mathbf{v}', \tau),$$

$$g_\varphi(\mathbf{v}, \tau = 0) = \delta(C)/4\pi C^2. \tag{B2}$$

Equation (B2) indicates that g_φ is a distribution function of electrons in a gas which is similar to hydrogen except that the cross sections for inelastic collisions in hydrogen have here become cross sections for absorption. The single electron is released into the system at time $\tau = 0$ and, under the influence of field and collisions, diffuses upwards in energy until it is absorbed. The sole dependence of the distribution on the time t', to which $\tau = 0$ corresponds, is through φ, the phase of the electric field at the time of release of the electron. After the electron has collided once or twice so that its direction of motion has been randomized, it is likely to lose all memory of the precise phase of the field at the time of its birth. That is, we expect that φ dependence of g_φ to damp out for $\tau > 1/\nu$.

If we consider the relation between $N(t, t')$ and $g(\mathbf{v}, t, t')$ given by Eq. (B2), it is evident that N depends only on the behavior of the electrons with energies above the inelastic threshold energy. The majority of these relatively high energy electrons will have suffered at least one collision since their birth at $\tau = 0$ with zero energy. Hence, we expect that φ dependence of N to be even weaker than that of g, leaving N to depend mainly on $\tau = (t - t')$. This expectation motivates the mathematics that follows.

One should note, however, that this reasoning does imply an upper limit on the field strength for which these conditions are valid. If the field strength is sufficiently high that the first collision is likely to be an inelastic one, then N itself will decay before its φ dependence has disappeared. In that case, we would not expect the φ independence of N to be useful.

We assume now that the time variation of g_φ can be factored into parts which vary slowly during a single period, $1/\omega$, and parts whose time variation is com-

parable to, or faster than that of the field. We are then able to write

$$g_\varphi(\mathbf{v}, \tau) = g_\varphi{}^0(\mathbf{v}, \tau) + g_\varphi{}^s(\mathbf{v}, \tau) \sin\omega\tau + g_\varphi{}^c(\mathbf{v}, \tau) \cos\omega\tau$$
$$+ g_\varphi{}^{2s}(\mathbf{v}, \tau) \sin2\omega\tau + g_\varphi{}^{2c}(\mathbf{v}, \tau) \cos2\omega\tau + \cdots. \tag{B3}$$

Such an expansion is reasonable only if all the coefficients of the trigonometric terms are slowly varying with respect to the field; otherwise there is no clear-cut way of deciding whether a part of the time dependence should be associated with one of the coefficients or with one of the trigonometric terms. This sets a lower limit on the frequency: $1/\omega$ must be small compared with the decay time of g, that is, small compared with an inelastic collision time.

Assuming that the frequency is high enough for this separation, we may substitute (B3) into (B2) and consider the equations obtained by equating the coefficients of like powers of $\exp(i\omega\tau)$. We shall satisfy these equations only up to and including the first power of the exponential, a procedure which is justified if variations in times shorter than $1/\omega$ are insignificant. This again implies the same lower limit on the frequency. The equations we must consider are

$$(\partial/\partial\tau + \nu)g^0$$

$$+ \tfrac{1}{2}a(\partial/\partial v_z)(\cos\varphi \, g^c - \sin\varphi \, g^s) = \nu_e N^0, \tag{B4a}$$

$$(\partial/\partial\tau + \nu)g^c + a \cos\varphi(\partial/\partial v_z)g^0 + \omega g^s = \nu_e N^c, \tag{B4b}$$

$$(\partial/\partial\tau + \nu)g^s - a \sin\varphi(\partial/\partial v_z)g^0 - \omega g^c = \nu_e N^s, \tag{B4c}$$

$$N^x = \frac{1}{4\pi} \int d\Omega \, g^x; \quad x = 0, c, s. \tag{B4d}$$

The task is now to obtain simple uncoupled equations for the angular averages N^x, since it is only these which are needed in (4.7). For the purpose of performing the angular average, it is convenient again to introduce the polar coordinates

$$|\mathbf{v}| = C,$$

$$v_z = \mu C,$$

$$\frac{\partial}{\partial v_z} = \mu \frac{\partial}{\partial C} + \frac{(1 - \mu^2)}{C} \frac{\partial}{\partial \mu},$$

and to define a current density Γ^0

$$\Gamma_\varphi{}^0(C, \tau) = \frac{1}{4\pi} \int d\Omega \, g_\varphi{}^0(\mathbf{v}, \tau)\mu C. \tag{B5}$$

Then the angular average of (B4b) and (B4c) may be converted, by means of an integration by parts, to

$$\left(\frac{\partial}{\partial\tau} + \nu - \nu_e\right)N^c + a \cos\varphi\left(\frac{\partial}{\partial C} + \frac{2}{C}\right)\frac{\Gamma^0}{C} + \omega N^s = 0, \tag{B6a}$$

$$\left(\frac{\partial}{\partial \tau}+\nu-\nu_e\right)N^s-a\,\sin\varphi\left(\frac{\partial}{\partial C}+\frac{2}{C}\right)\frac{\Gamma^0}{C}-\omega N^c=0. \quad \text{(B6b)}$$

Consider $\Gamma_\varphi{}^0(C,\tau)$, the slowly varying part of the current at time τ carried by electrons of speed C. As a current—a vector quantity—it must be identically zero unless there is some spatial direction which is of physical significance. There are only two spatial directions, however, on which g_φ can possibly depend. One of these is the direction of the electric field at $\tau=0$, when the electron was released, and the other is the electric field at τ, the present. On the time scale to which Γ^0 corresponds, the high-frequency field at τ can establish no directional preference because $\Gamma_\varphi{}^0(C,\tau)$ is nearly equal to $\Gamma_\varphi{}^0(C,\ \tau-\pi/\omega)$ by reason of the slow variation of g^0, while the field at the earlier time is directed opposite to itself at the later time. This leaves only the field direction at $\tau=0$ as a significant direction, but once the electron loses memory of the initial phase, then this direction too has no special significance. Hence, there is no preferred direction after times greater than $1/\nu$ and Γ^0 must vanish.

The vanishing of Γ^0 makes it possible to solve Eqs. (B6). We find

$$N^c=\mp iN^s=\text{const }\exp[(\nu_e-\nu\pm i\omega)\tau]. \quad \text{(B7)}$$

We have explicitly taken N^c and N^s to be slowly varying compared with the field variation, while the solution above states just as explicitly that they oscillate at frequency ω. We must, therefore, take the constant in Eq. (B7) to be zero, so that N^c and N^s also vanish identically and can, therefore, be deleted from (B4).

The Laplace transform of the Eq. (B4), with transforms denoted by tildes and initial values by G, is

$$(p+\nu)\tilde{g}^0+\tfrac{1}{2}a(\partial/\partial v_z)(\cos\varphi\,\tilde{g}^c-\sin\varphi\,\tilde{g}^s)=\nu Q/(4\pi), \quad \text{(B8a)}$$

$$(p+\nu)\tilde{g}^c+\omega\tilde{g}^s=G^c-a\cos\varphi\,\partial\tilde{g}^0/\partial v_z, \quad \text{(B8b)}$$

$$(p+\nu)\tilde{g}^s-\omega\tilde{g}^c=G^s+a\sin\varphi\,\partial\tilde{g}^0/\partial v_z, \quad \text{(B8c)}$$

$$Q=\frac{4\pi}{\nu}G^0+r_e\int d\Omega\,\tilde{g}^0. \quad \text{(B8d)}$$

Equation (B8b,c) may be solved for g^s and g^c in terms of the quantities on the right-hand side of the equation. From the solution, we construct a term appearing in (B8a)

$$\cos\varphi\,\tilde{g}^c-\sin\varphi\,\tilde{g}^s=-A\,(\partial/\partial v_z)\tilde{g}^0+B(\mathbf{v},\varphi); \quad \text{(B9a)}$$

$$A=\frac{a(p+\nu)}{1+\omega^2/(p+\nu)^2}, \quad \text{(B9b)}$$

$$B=\frac{[(p+\nu)G^c-\omega G^s]\cos\varphi}{(p+\nu)^2+\omega^2}$$
$$-\frac{[(p+\nu)G^s+\omega G^c]\sin\varphi}{(p+\nu)^2+\omega^2}. \quad \text{(B9c)}$$

The functions $G^s(\mathbf{v})$ and $G^c(\mathbf{v})$ are the, as yet unspecified, initial values of $g^s(\mathbf{v},\tau=0)$ and $g^c(\mathbf{v},\tau=0)$. The choice of these initial values will be deferred till the end. For the present, let us define the effective acceleration

$$a_e{}^2=\frac{a^2}{2[1+\omega^2/(p+\nu)^2]}. \quad \text{(B10)}$$

Then inserting (B9) into (B8a) yields

$$\left[1+\frac{a_e{}^2}{(p+\nu)^2}\frac{\partial^2}{\partial v_z{}^2}\right]g^0=\frac{1}{(p+\nu)}\left[\frac{\nu}{4\pi}Q+\frac{a}{2}\frac{\partial B}{\partial v_z}\right]. \quad \text{(B11)}$$

The Green's function K for inverting the differential operator on the left of (B11) is given by

$$K(v_z,w)=\frac{1}{2}\left(\frac{p+\nu}{a_e}\right)e^{-|v_z-w|(p+\nu)/a_e}.$$

Using it, we integrate Eq. (B11) to obtain

$$\tilde{g}_\varphi{}^0(\mathbf{v},p)=\frac{1}{2a_e}\int_{-\infty}^{\infty}dw$$

$$\times\left[\frac{\nu}{4\pi}Q(C')+\frac{a}{2}\frac{\partial}{\partial w}B(\mathbf{v'},\varphi)\right]e^{-|v_z-w|(p+\nu)/a_e};$$

$$C'=|\mathbf{v'}|,$$

$$\mathbf{v'}=(v_x,v_y,w).$$

The region of integration may be split at $w=v_z$ and the second term integrated by parts. On introducing a variable

$$R=|v_z-w|=|\mu C-w|,$$

we obtain

$$\tilde{g}_\varphi{}^0(\mathbf{v},p)=\frac{1}{2a_e}\int_0^\infty dR\,e^{-R(p+\nu)/a_e}\left[\frac{\nu}{4\pi}Q(C')+\frac{\nu}{4\pi}Q(C'')\right.$$

$$\left.-(p+\nu)B(\mathbf{v'},\varphi)+(p+\nu)B(\mathbf{v''},\varphi)\right]; \quad \text{(B12a)}$$

$$\mathbf{v'}=(v_x,\,v_y,\,\mu C-R);$$

$$C'=|\mathbf{v'}|=(C^2-2\mu CR+R^2)^{1/2}, \quad \text{(B12b)}$$

$$\mathbf{v''}=(v_x,\,v_y,\,\mu C+R);$$

$$C''=|\mathbf{v''}|=(C^2+2\mu CR+R^2)^{1/2}. \quad \text{(B12c)}$$

Now let us construct $\tilde{\Gamma}^0$, the Laplace transform of the Γ^0 defined in (B5), by multiplying (B12a) by $\mu C/4\pi$ and integrating over angle. A change of integration variable from μ to $-\mu$ in those terms which depend on C'' or v''

causes the terms in Q to drop out, leaving

$$\frac{\tilde{\Gamma}_\varphi{}^0(C,p)}{C} = -\frac{(p+\nu)}{4\pi a_e} \int_0^\infty dR \int d\Omega$$

$$\times \mu e^{-R(p+\nu)/a_e}[B(v_1,\varphi)+B(v_2,\varphi)];$$

$$v_1 = (v_x, v_y, \mu C - R), \qquad (\text{B13})$$

$$v_2 = (v_x, v_y, R - \mu C).$$

If we were to substitute (B9c) into (B13), it would become evident that $\tilde{\Gamma}^0$ would change sign if the phase φ were altered by π radians. This indicates that $\tilde{\Gamma}^0$ contains, at the most, a memory of the initial phase but no component that results from the instantaneous value of the field. This is gratifying, because it is consistent with the assumption that the field varies too rapidly to establish a spatially preferred direction to which the relatively sluggish g^0 could respond.

Since the entire development so far followed from the assumption that Γ^0 in (B6) could be ignored, we must choose the initial functions G^s and G^c so that $\tilde{\Gamma}^0$ in (B13) also vanishes. The simplest choice is to take G^s and G^c both zero. Then $\tilde{\Gamma}^0$ vanishes because B vanishes. This leaves only G^0 available for satisfying the initial condition (B2). The initial condition requires, then, that

$$G^0 = \delta(C)/(4\pi C^2). \qquad (\text{B14})$$

We may now integrate (B12a) over angle and, inserting (B14) into (B8d) we obtain the pair of equations:

$$\int d\Omega\ g^0 = \frac{\nu}{2a_e} \int_{-1}^{1} d\mu \int_0^\infty dR\ e^{-R(p+\nu)/a_e}Q(C'), \qquad (\text{B15a})$$

$$Q(C') = \delta(C)/(\nu C^2) + r_e \int d\Omega\ \tilde{g}^0. \qquad (\text{B15b})$$

Note that to this order of approximation, there is no φ dependence in the angular integral of g^0. We already found that the angular integrals of g^c and g^s vanish; this means that to this order of approximation, the angular integral of $g_\varphi(v,\tau)$ itself also is independent of φ. It follows that the function $N(t,t')$ defined by (4.5) is really a function of the single variable $\tau = t - t'$, rather than of t and t' separately. Hence, we may take the Laplace transform leading from Eqs. (4.4) and (4.5) to Eqs. (4.6) and (4.7). Observe now the identity between Eqs. (B15) and (3.16b)–(18), between (4.6) and (3.8), and between (4.7) and (3.9). These identities are sufficient to establish that the density in the microwave avalanche breakdown increases exponentially like $\exp(\beta t)$ and that β depends on the effective acceleration

$$a_e = \frac{a/\sqrt{2}}{[1+\omega^2/(\nu+\beta)^2]^{1/2}}, \qquad (\text{B16})$$

in exactly the same way as the dc-growth rate β depends on the dc acceleration provided that $\omega > \beta$.

The equivalence between the microwave and dc cases goes further than just the growth rates, of course. A comparison of Eqs. (B12) and (3.17) reveals that

$$\tilde{g}^0(\mathbf{v},\beta) = \tfrac{1}{2}[g_{dc}(\mathbf{v},\beta) + \tilde{g}_{dc}(\mathbf{V}',\beta)], \qquad (\text{B17})$$

where

$$\mathbf{V}' = (v_x, v_y, -v_z).$$

This result is to be expected because the slowly varying g^0 in the microwave treatment is unable to distinguish between the two directions in which the field can point.